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A Pseudocompact Completely Regular Frame which is not Spatial

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Abstract Compact regular frames are always spatial. In this note we present a method for constructing non-spatial frames. As an application we show that there is a countably compact (and hence pseudocompact) completely regular frame which is not spatial.

Keywords Frame · Non-spatial frame · Countably compact · Pseudocompact · $\beta\omega$ · Čech-Stone compactification

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1 Introduction

It is known that every compact regular frame is spatial. In fact, this is equivalent to the Boolean Ultrafilter Theorem, which states that every proper filter in a Boolean algebra is contained in an ultrafilter (see, for instance, [1]). During BBFest2011, a conference held at the University of Cape Town from 28 to 30 March 2011 on the occasion of the 85th birthday of Bernhard Banaschewski, a question was asked if every completely regular pseudocompact frame is spatial.

In this note we show that, in contrast to the compact regular case, there is a non-spatial completely regular countably compact (and hence pseudocompact) frame. A usual source of non-spatial frames is the class of Boolean frames without atoms. However, a completely regular non-spatial countably compact frame cannot be among the Boolean ones because Boolean frames are paracompact [9], and paracompact pseudocompact regular frames are compact [4].

Not only is the example we proffer non-spatial, it in fact has no points whatsoever.

For our construction we first describe a simple method for constructing new frames from spatial ones. It is then applied for the construction of the example that we just mentioned showing that the method has potential. Moreover, our arguments are very set theoretical. We hope that there will be more applications of our set theoretical method to the theory of frames.

2 Preliminaries

Our reference for frames is Picado and Pultr [8]. Throughout the paper, by frame we shall mean a completely regular frame. We denote the frame of open sets of a topological space X by $\mathfrak{O}X$. A frame is *spatial* if it is isomorphic to $\mathfrak{O}X$, for some topological space X. An internal characterisation of spatial frames is that every element can be written as a meet of *meet-irreducible* elements. These are the elements p such that $p < 1$, and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$.

We use the notation of [2] regarding the ring $\mathcal{R}L$ of real-valued continuous functions on a frame L. An element α of $\mathcal{R}L$ is said to be *bounded* if there exist $p, q \in \mathbb{Q}$ such that $\alpha(p, q) = 1_{\mathfrak{L}(\mathbb{R})}$. A frame L is *pseudocompact* if $\mathcal{R}L = \mathcal{R}^*L$; where the latter designates the subring consisting of bounded elements. There are several internal characterisations of pseudocompact frames (see e.g. [3] and [5]). For example, a completely regular frame is pseudocompact if and only if every countable cover has a finite subset with dense join. A frame is *countably compact* if every countable cover has a finite subcover. Clearly, a countably compact frame is pseudocompact.

3 A Method for Constructing New Frames from Spatial Ones

All topological spaces under discussion are Tychonoff. For standard facts about cardinal functions in topology, see Juhász [7]. If A and B are sets, then $A \triangle B$ denotes their symmetric difference.

In this section we will describe a simple method for constructing new frames from frames of the form $\mathfrak{O}X$. To this end, let κ be an infinite cardinal number, X a space of weight at most κ. This means that X has a basis for its topology of cardinality at most κ. Moreover, let $\mathcal{I}$ be a κ^+-*complete* ideal of subsets of X. This means that $\mathcal{I}$ is an ideal of subsets of X which has the following property: if $\mathcal{A} \subseteq \mathcal{I}$ and $|\mathcal{A}| \leq \kappa$, then $\bigcup \mathcal{A} \in \mathcal{I}$.

Lemma 3.1 *If $\mathcal{A} \subseteq \mathfrak{O}X$ then there exists $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \kappa$ and $\bigcup \mathcal{A} = \bigcup \mathcal{B}$.*

Proof Obvious from the fact that the weight of X is at most κ. □

Now let $L = \mathfrak{O}X$. We define a relation $\sqsubseteq$ on L as follows: for $U, V \in L$ we put

$$U \sqsubseteq V \quad \text{iff} \quad U \setminus V \in \mathcal{I}.$$

Lemma 3.2 *If $U, V, W \in L$, then*

(1) $U \sqsubseteq U$,
(2) *if* $U \sqsubseteq V$ *and* $V \sqsubseteq W$, *then* $U \sqsubseteq W$,
(3) *if* $U \sqsubseteq V$ *and* $U \sqsubseteq W$, *then* $U \sqsubseteq V \cap W$,

Proof For (1) and (2) observe that $U \setminus U = \emptyset \in \mathcal{I}$ and

$$U \setminus W \subseteq (V \setminus W) \cup (U \setminus V) \in \mathcal{I}.$$

For (3), observe that $U \setminus V \in \mathcal{I}$ and $U \setminus W \in \mathcal{I}$. Hence $U \setminus (V \cap W) \in \mathcal{I}$ since

$$U \setminus (V \cap W) \subseteq (U \setminus V) \cup (V \setminus W).$$

So we are done. □

Next, we define an equivalence relation $\sim$ on L by

$$U \sim V \quad \text{iff} \quad U \sqsubseteq V \text{ and } V \sqsubseteq U.$$

Thus,

$$U \sim V \quad \text{iff} \quad U \triangle V \in \mathcal{I}.$$

For $U \in L$ we let $[U]$ denote its $\sim$-equivalence class. Now put $M = L/{\sim}$, and define a partial order $\leq$ on M by

$$[U] \leq [V] \quad \text{iff} \quad U \sqsubseteq V.$$

It is clear that this definition is well-defined.

We will show that M is a frame.

Lemma 3.3 *If $[U], [V] \in M$, then $[U] \wedge [V]$ exists and is equal to $[U \cap V]$.*

Proof It is clear that $[U \cap V] \leq [U], [V]$. Take an element $O \in L$ such that $[O] \leq [U], [V]$. Then $[O] \leq [U \cap V]$ by Lemma 3.2(3). Hence $[U] \wedge [V] = [U \cap V]$. □

Lemma 3.4 *If $[U_i] \in M$ for every $i \in I$, then $\bigvee_{i\in I}[U_i]$ exists and is equal to $\left[\bigcup_{i\in I} U_i\right]$.*

Proof Clearly $[U_j] \leq \left[\bigcup_{i\in I} U_i\right]$ for every $j \in I$. Now assume that for some $[O] \in M$ we have that $[U_i] \leq [O]$ for every $i \in I$. By Lemma 3.1, there is a subset I' of I of cardinality at most κ such that

$$\bigcup_{i\in I'} U_i = \bigcup_{i\in I} U_i.$$

Hence, since $U_i \setminus O \in \mathcal{I}$ for every $i \in I'$ and $\mathcal{I}$ is κ^+-complete, we have that

$$\left(\bigcup_{i\in I} U_i\right) \setminus O = \left(\bigcup_{i\in I'} U_i\right) \setminus O = \bigcup_{i\in I'} (U_i \setminus O) \in \mathcal{I}.$$

But then $\left[\bigcup_{i\in I} U_i\right] \leq [O]$. Hence $\bigvee_{i\in I}[U_i] = \left[\bigcup_{i\in I} U_i\right]$. □

So now it is indeed easy to see that M is a frame with bottom $[\emptyset] = \{U \in \mathfrak{O}X : U \in \mathcal{I}\}$ and top $[X] = \{U \in \mathfrak{O}X : X \setminus U \in \mathcal{I}\}$.

Theorem 3.5 *M is a completely regular frame.*

Proof Observe that by Lemmas 3.3 and 3.4 the function $f\colon L \to M$ defined by $f(U) = [U]$ is a surjective frame homomorphism. Thus, M is a quotient of the completely regular frame L, and is therefore completely regular. □

Remark 3.6 Observe that this method is very general since no specification was made for the ideal $\mathcal{I}$ except for the fact that it is κ^+-complete. It is also possible to consider τ^+-complete ideals, where τ is some cardinal number less than κ, and vary the construction a little.

4 The Example

We now describe the promised example of a countably compact completely regular frame which is not spatial. Let $\mathfrak{c}$ denote the cardinality of the continuum.

Let X be a compact Hausdorff space with the following properties: the weight of X is $\mathfrak{c}$ and if $A \subseteq X$ is closed, then either A is finite or $|A| = 2^{\mathfrak{c}}$. Let $\mathcal{I}$ be the ideal of subsets of X of cardinality at most $\mathfrak{c}$. Then clearly $\mathcal{I}$ is $\mathfrak{c}^+$-complete. We will show that the frame M that was constructed in Section 3 from the frame $\mathfrak{O}X$ is completely regular, countably compact but not spatial.

An example of such a space X is $\beta\omega \setminus \omega$, the Čech-Stone remainder of the countable discrete space ω (see for example [6, 9.12] and [10, 3.17].) Moreover, X has no isolated points ([6, 9.5]; see also [10, 3.12]). This is just one example of a space with the required properties, there are many others.

Since a frame is spatial if and only if each of its elements is the meet of meet-irreducible elements (see [8, 5.3]), the following result implies that M is not spatial.

Lemma 4.1 *M has no meet-irreducible elements.*

Proof Assume that $[U] \neq [X]$. Then $X \setminus U$ has cardinality at least $\mathfrak{c}^+$. Let $\mathcal{E}$ be the collection of all relatively open subsets of $X \setminus U$ of cardinality at most $\mathfrak{c}$.

Claim $|\bigcup \mathcal{E}| \leq \mathfrak{c}$.

Observe that the weight of $X \setminus U$ is at most $\mathfrak{c}$. Hence there is a subcollection $\mathcal{F}$ of $\mathcal{E}$ such that $|\mathcal{F}| \leq \mathfrak{c}$ and $\bigcup \mathcal{F} = \bigcup \mathcal{E}$ (Lemma 3.1). Hence the Claim follows.

Since $X \setminus U$ has cardinality at least $\mathfrak{c}^+$, there are distinct $p, q \in (X \setminus U) \setminus \bigcup \mathcal{E}$. Let A and B be disjoint open neighborhoods in X of p respectively q. Then by Lemma 3.3,

$$[A \cup U] \wedge [B \cup U] = [(A \cup U) \cap (B \cup U)] = [U]. \tag{1}$$

However, $A \cap (X \setminus U)$ has cardinality at least $\mathfrak{c}^+$ since otherwise $A \cap (X \setminus U) \in \mathcal{E}$ which violates $p \notin \bigcup \mathcal{E}$. Hence $[A \cup U] \nleq [U]$, and similarly, $[B \cup U] \nleq [U]$. From this and Eq. 1 it follows that $[U]$ is not meet-irreducible. □

Theorem 4.2 *The frame M is completely regular, countably compact but not spatial.*

Proof By Lemma 4.1 we only need to prove that M is countably compact. To this end, let $[U_n] \in M$ for $n < \omega$ such that $\bigvee_{n<\omega} [U_n] = [X]$. Put $U = \bigcup_{n<\omega} U_n$. Then by Lemma 3.4 $[U] = [X]$, which implies that $X \setminus U$ has cardinality at most $\mathfrak{c}$. We claim that there exists $N < \omega$ such that $U = \bigcup_{i \leq N} U_i$. Striving for a contradiction, assume that this is not true. Then we can find an increasing sequence of integers

$$N_0 < N_1 < \cdots < N_n < \cdots$$

and elements $x_n \in U_{N_n} \setminus \bigcup_{m<n} U_{N_m}$ for $n < \omega$. Put $A = \overline{\{x_n : n < \omega\}}$. Since $A \cap U_n$ is finite for every n and is contained in $\{x_n : n < \omega\}$, we have that $B = A \setminus \{x_n : n < \omega\}$ is contained in $X \setminus U$. But A is not finite, hence has cardinality $2^{\mathfrak{c}}$. Therefore B has cardinality $2^{\mathfrak{c}}$, which is a contradiction since $X \setminus U$ has cardinality at most $\mathfrak{c}$.

Thus, $[U] = [X]$, from which it follows that $[U_0] \vee \cdots \vee [U_N] = [X]$. □

Remark 4.3 By Picado and Pultr [8, 6.3.4] each locally compact frame is spatial. Hence M is not locally compact. It may be illustrative to give a direct argument that M is not compact for the case that $X = \beta\omega \setminus \omega$. In fact, the argument gives us that M is not Lindelöf. To see this, pick an arbitrary $x \in X$, and let $\mathcal{C}$ be the collection of all clopen subsets of X that do not contain x. Since X is zero-dimensional, by Lemma 3.4 it is clear that $\bigvee_{C \in \mathcal{C}} [C] = [X]$. Assume that there is a countable $\mathcal{F} \subseteq \mathcal{C}$ such that $\bigvee_{C \in \mathcal{F}} [C] = [X]$. Then, on the one hand, $X \setminus \bigcup \mathcal{F}$ is a nonempty closed G_δ-subset of X since it contains x, and on the other hand must have cardinality at most $\mathfrak{c}$. But every nonempty G_δ-subset of X has cardinality $2^{\mathfrak{c}}$ [10, 3.27 and 3.11] and so this is a contradiction.

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